Inspired by Game Theory, solved by complexity: an innocent problem with deep connections

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- Objectives
- Importance of this work
- Story

Preliminaries

Matrix Games

Matrix games.

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} & & \\ & m_{i,j} & \\ & & \end{pmatrix} \end{array}$$

Strategies.

$$p \in \Delta[m] \qquad q \in \Delta[m]\,.$$

Value.

$$\mathrm{val} M := \max_{p \in \Delta[m]} \min_{q \in \Delta[n]} p^\top M q\,.$$

Perturbed Matrix Games

Polynomial matrix games. Matrix games where payoff entries are given by polynomials.

$$M(\varepsilon) = M_0 + M_1 \varepsilon + \ldots + M_K \varepsilon^K .$$

Value function.

$$\varepsilon \mapsto \mathrm{val} M(\varepsilon) .$$

Example

Consider $\varepsilon > 0$.

$$M(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} \varepsilon \,.$$

The optimal strategy is given by, for $\varepsilon < 1/2$,

$$p^*_\varepsilon = \left(\frac{1+\varepsilon}{2+3\varepsilon}, \frac{1+2\varepsilon}{2+3\varepsilon} \right)^\top \,.$$

Therefore,

$$\mathrm{val} M(\varepsilon) = \frac{\varepsilon^2}{2+3\varepsilon} \,.$$

Example 2

Consider $\varepsilon > 0$.

$$M(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 0 & -2 \end{pmatrix} \varepsilon \,.$$

The optimal strategy is given by, for $\varepsilon < 2/3$,

$$p^*_\varepsilon = \left(\frac{1-\varepsilon}{2-3\varepsilon}, \frac{1-2\varepsilon}{2-3\varepsilon} \right)^\top \,.$$

Therefore,

$$\mathrm{val} M(\varepsilon) = \frac{\varepsilon^2}{2-3\varepsilon} \,.$$

Questions

Questions

Definition (Value-positivity problem)

Is the value function increasing?
$\exists \varepsilon_0 > 0$ such that $\forall \varepsilon \in [0, \varepsilon_0] \quad \mathrm{val} M(\varepsilon) \geq \mathrm{val} M(0)$.

Definition (Functional form problem)

What is the value function?
Return the maps $\mathrm{val} M(\cdot)$ and $p^*(\cdot)$, for $\varepsilon \in [0, \varepsilon_0]$.

Definition (Uniform value-positivity problem)

Can th max-player guarantee at least $\mathrm{val} M(0)$ with a fixed strategy?
$\exists p_0 \in \Delta[m] \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in [0, \varepsilon_0] \quad \mathrm{val}(M(\varepsilon); p_0) \geq \mathrm{val} M(0)$.

Results

Algorithms

Lemma (Poly-time algorithms)

When data is rational, there are polynomial-time algorithms for all three problems.

Main ideas

Value-positivity and functional form.
$\varepsilon \mapsto \mathrm{val} M(\varepsilon)$ is rational and have coefficients that are at most exponential.

Main ideas: Uniform value-positivity

LP solution of Matrix Games.

$$(P_M)\begin{cases} \max_{p,z} & z & & \\ s.t. & (p^\top M)_j & \geq z & \forall j \in [m] \\ & p & \in \Delta([m]) & \end{cases}$$

Leading coefficients of a strategy. For a fixed strategy p, we can think about the leading coefficients against every column action

$$\begin{array}{c} \\ M_0 \\ M_1 \\ M_2 \\ M_3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \boxed{2} & 0 \\ 0 & -1 & \boxed{-1} \\ \boxed{0} & 0 & 0 \end{pmatrix} \end{array}$$

Consequences

Linear Programming

An LP is the following optimization problem.

$$(P)\begin{cases} \min_x & c^\top x & \\ s.t. & Ax & \leq b \\ & x & \geq 0\,, \end{cases}$$

LPs with errors

An LP with errors is the following family of optimization problems.

$$(P_\varepsilon)\begin{cases} \min_x & c(\varepsilon)^\top x \\ s.t. & A(\varepsilon)x \leq b(\varepsilon) \\ & x \geq 0\,, \end{cases}$$

Examples

$$(P_\varepsilon)\begin{cases} \min_x & x \\ s.t. & x \leq -\varepsilon \\ & -x \leq -\varepsilon\,. \end{cases}$$

Examples 2

$$(P_\varepsilon)\begin{cases} \max_{x,y} & x+y \\ s.t. & x \leq 0 \\ & y+\varepsilon x \leq 0\,. \end{cases}$$

For $\varepsilon < 1$,

$$\text{val}(P_\varepsilon) \equiv 0$$
$$(x,y)^*(\varepsilon) \equiv (0,0)\,.$$

Sub-class of LPs

Definition (A priori bounded)

The Lp with errors (P_ε) is a priori bounded if both the primal and dual are uniformely bounded for ε small enough.

Questions

Definition (Weakly robust)

Is there a solution?

$\exists \varepsilon_0 > 0$ such that, $\forall \varepsilon \in [0, \varepsilon_0] \quad (P_\varepsilon)$ is feasible.

Definition (Functional form)

What is the solution?

The maps $\text{val}(P_\cdot)$ and $x^*(\cdot)$, for $\varepsilon \in [0, \varepsilon_0]$.

Definition (Strongly robust)

Is there a constant solution?

$\exists x^* \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in [0, \varepsilon_0], \quad x^*$ is also a solution of (P_ε).

Reminder: Equivalence between Matrix Games and LPs

Theorem (Adler03)

Matrix games and LPs a poly-time equivalent.

- [Dantzig51] gives an incomplete proof.
- The reduction depends on the computational model: rational, algebraic or real data.

Results

Lemma (LP with error to polynomial matrix games)

There is a polynomial-time reduction from robustness problems to the respective value-positivity problem, which preserves the degree of the error perturbation, for algebraic data.

Stochastic Games

Matrix games.

$$i \begin{pmatrix} & \overset{j}{} & \\ & (m_{i,j}, \rightarrow) & \end{pmatrix} \qquad i \begin{pmatrix} & \overset{j}{} & \\ & (m_{i,j}, \leftarrow) & \end{pmatrix}$$

Strategies.

$$p \in (\Delta[m])^n \qquad q \in (\Delta[m])^n .$$

Discounted and limit value. For $\lambda \in (0,1)$,

$$\operatorname{val}_\lambda M := \max_p \min_q \lambda \sum_{i \geq 1} (1-\lambda)^i \left(p_i^\top M^{(i)} q_i \right).$$

$$\operatorname{val} M := \lim_{\lambda \to 0^+} \operatorname{val}_\lambda M .$$

Stochastic Games and Matrix Games

Theorem

Consider a Stochastic Game Γ. There exists a parametrized polynomial matrix game

$$M_z = N(\lambda) - z\tilde{N}(\lambda)\,,$$

where $N, \tilde{N}$ are Matrix Games, such that, for all $z \in \mathbb{R}$ and $\lambda \in (0,1)$,

$$\text{val} M_z(\lambda) \geq 0 \qquad \Leftrightarrow \qquad \text{val}_\lambda \Gamma \geq z\,.$$

Value-positivity for Stochastic Games

Consider a Stochastic game Γ and its parametrized polynomial matrix game $(M_z)_z$.

Lemma (Value-positivity)

For all $z \in \mathbb{R}$, if M_z is value-positive, then, for all λ sufficiently small,

$$\text{val}_\lambda \Gamma \geq z \,.$$

Lemma (Uniform value-positivity)

For all $z \in \mathbb{R}$, if M_z is uniform value-positive, then there exists a fixed strategy $p \in (\Delta[m])^n$ such that, for all λ sufficiently small,

$$\text{val}_\lambda(\Gamma; p) \geq z \,.$$

Do you have players in your problem?